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A Polynomial-Time Algorithm for Affine Variational Inequalities

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Abstract. This short note illustrates that the recently proposed polynomial-time algorithm for convex quadratic programs can easily be extended to solve affine variational inequality problems $\mathrm{VI}(X, F)$ where the mapping F is monotone. A small numerical example is used to illustrate the convergence properties of this new algorithm.

In the past few years, there has been an explosion of interest in interior-point algorithms for linear programs due to their nice worst-case performance. In recent years, these algorithms have been extended to solve convex quadratic programs and linear complementarity problems defined by a positive semidefinite matrix M [1–4]. Kojima *et al.* [5] have also extended their algorithm to the case of the nonlinear complementarity problem with a uniform P-function.

As shown by the recent survey article on finite-dimensional variational inequalities [6], there has been no work on such polynomial-time algorithms for the general variational inequality problem. Lüthi [7] extended Khachiyan's ellipsoid algorithm to solve the general variational inequality problem, but did not establish a polynomial complexity result. The purpose of this note is to show that Monteiro and Adler's [3] algorithm for convex quadratic programs can be used to solve a variational inequality defined by an affine, monotone mapping. The formulation of the problem will be given in the next section, Section 2 briefly describes the algorithm, and the last section will illustrate the performance of this algorithm on a small numerical example.

1. PROBLEM FORMULATION

Following the notation in [6], the affine variational inequality problem $\mathrm{VI}(\Omega, q, M)$ is to find a vector $x^* \in \Omega$ such that

$$(q + M\,x^*)^T (x - x^*) \geq 0, \qquad \forall x \in \Omega, \tag{1}$$

where

$$\Omega = \{x \in \Re^n : A\,x = b, \quad x \geq 0\},$$

and $A \in \Re^{m \times n}$, $b \in \Re^m$. When the matrix M is symmetric, problem (1) is equivalent to the first-order conditions of the following quadratic program:

$$\min_{x \in \Omega} q^T\,x + \frac{1}{2} x^T\,M\,x.$$

In such a case, the algorithm of Monteiro and Adler can be immediately applied. When M is asymmetric, however, this equivalence is destroyed.

The variational inequality (1) can be converted into a mixed complementarity problem [6] through the use of generalized Karush-Kuhn-Tucker (KKT) conditions. Letting $y \in \Re^m$ denote

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the generalized KKT multipliers for the constraints $A\,x = b$, the mixed complementarity problem is defined by the following set of conditions:

$$q + M\,x - A^T\,y \;\geq\; 0, \quad x \geq 0, \quad [q + M\,x - A^T\,y]^T x = 0, \tag{2}$$
$$A\,x \;=\; b. \tag{3}$$

In the spirit of Monterio and Adler [3], let us define a barrier function for the VI(Ω, F) as $(-\mu\,x_i^{-1})$ for each $i = 1, 2, \ldots, n$ where $\mu > 0$ is a constant; this function is simply the derivative of the logarithmic barrier function used in [3]. The augmented variational inequality VI(Ω_0, F) is defined as:

$$(q + M\,x^* - \mu(X^*)^{-1}\,e)^T(x - x^*) \geq 0, \qquad \forall x \in \Omega_0, \tag{4}$$

where

$$\begin{aligned} \Omega_0 &= \{x \in \Re^n : A\,x = b, \quad x > 0\}, \\ F(x) &= q + M\,x^* - \mu(X^*)^{-1}\,e, \end{aligned}$$

X^* is a diagonal matrix with entries x_i^*, and e is an n-dimensional column vector of ones. Writing the generalized KKT conditions for this problem, one has:

$$q + M\,x - A^T\,y - \mu\,X^{-1}\,e \;=\; 0, \qquad x > 0, \tag{5}$$
$$A\,x \;=\; b. \tag{6}$$

Letting $z_i = q_i + (M\,x - A^T\,y)_i$, $z = (z_1, z_2, \ldots, z_n)^T$, and $Z = \mathrm{diag}(z_i)$, the above system of equations can be rewritten as:

$$Z\,X\,e \;=\; \mu\,e, \tag{7}$$
$$z - M\,x + A^T\,y \;=\; q, \qquad x > 0,\, y > 0, \tag{8}$$
$$A\,x \;=\; b. \tag{9}$$

One will immediately notice that the system of equations (7)–(9) is identical to the system analyzed in Part II of [3] except for the fact that the matrix M is asymmetric in general.

The key question in the analysis of (7)–(9) is whether or not the solution to this system is unique and thus, defines a unique path $w(\mu) = (x(\mu), y(\mu), z(\mu))$, for $\mu > 0$. In [3], this uniqueness is argued using quadratic programming duality theory. In the context of the affine variational inequality problem, one can easily establish the following result due to the fact that the mapping $F(x) = q + M\,x - \mu\,X^{-1}\,e$ is strongly monotone for all $\mu > 0$; the argument follows directly from the application of the ideas presented in [3] and Corollary 3.2 in [6]:

PROPOSITION 1. *If*

(i) M is positive semidefinite (F is a monotone mapping),

(ii) the matrix A is of full rank ($\mathrm{rank}(A) = m$),

(iii) the set $T = \{w = (x, y, z) \in \Re^n \times \Re^m \times \Re^n : z - M\,x + A^T\,y = q, \quad z > 0\}$ is nonempty, and

(iv) the set Ω_0 is nonempty,

then there exists a unique solution $w(\mu) = (x(\mu), y(\mu), z(\mu))$ to the system (7)–(9) for every $\mu > 0$, if the original problem VI(Ω, q, M) has a nonempty, bounded solution set.

The paper by Harker and Pang [6] provides a survey of the numerous conditions on the problem VI(X, F) which ensure that its solution set is nonempty.

2. THE ALGORITHM

Having established Proposition 1, a careful reading of [3] uncovers the fact that the symmetry of M in the quadratic programming case is never used to establish the convergence and polynomial complexity of the basic algorithm. Therefore, the exact same algorithm and initial solution as defined in Part II of [3] can be used to compute a solution to $\mathrm{VI}(\Omega, q, M)$. In particular, the complexity of this algorithm is $O(n^3L)$ where L is the size of the input data.

3. NUMERICAL EXAMPLE AND CONCLUSION

In order to illustrate this algorithm on a small test problem, consider the following example from [8]. The set Ω is defined by $\Omega = \{x \in \Re^5_+ : A\,x = b\}$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

and $b = (210, 120)^T$. The function $F(x) = q + M\,x$ is given by

$$M = \begin{pmatrix} 10 & 0 & 0 & 5 & 0 \\ 0 & 15 & 0 & 0 & 5 \\ 0 & 0 & 20 & 0 & 0 \\ 2 & 0 & 0 & 20 & 0 \\ 0 & 1 & 0 & 0 & 25 \end{pmatrix},$$

and $q = (1000, 950, 3000, 1000, 1300)^T$. It is easy to verify that M is positive definite and that the unique solution of this variational inequality problem is $x^* = (120, 90, 0, 70, 50)^T$. Using the artificial problem defined in Section 6 of [3, Part II] with $\alpha = 2^{40}$ and $\lambda = 2^{20}$, the sequence listed in Table 1 was generated. This example illustrates the fact that the algorithm tends to converge rapidly, particularly as it approaches the solution point.

Table 1. Generated sequence for the example.

Iteration	μ	x_1	x_2	x_3	x_4	x_5
0	1.00×10^{18}	1.00×10^6	1.00×10^6	1.00×10^6	1.00×10^6	1.00×10^6
100	1.03×10^{16}	6.13×10^4	6.13×10^4	6.13×10^4	6.13×10^4	6.13×10^4
200	1.06×10^{14}	653.0	653.0	653.0	653.0	653.0
300	1.09×10^{12}	71.2	71.2	71.2	61.2	61.2
400	1.13×10^{10}	70.0	70.0	70.0	60.0	60.0
500	1.16×10^8	70.0	70.0	69.9	60.0	60.0
600	1.20×10^6	73.4	72.1	64.5	60.7	59.3
700	1.23×10^4	111.0	85.5	13.4	68.9	51.1
800	1.27×10^2	120.0	89.9	2.77×10^{-1}	70.0	50.0
900	1.31×10^0	120.0	90.0	2.91×10^{-3}	70.0	50.0
1000	1.35×10^{-2}	120.0	90.0	3.00×10^{-5}	70.0	50.0
1100	1.39×10^{-4}	120.0	90.0	3.09×10^{-7}	70.0	50.0
1200	1.43×10^{-6}	120.0	90.0	3.18×10^{-9}	70.0	50.0

Future research will be devoted to the extension of the concepts discussed in this paper to the problem of solving $\mathrm{VI}(\Omega, F)$ when F is a nonlinear mapping akin to the work reported in [5], and the efficient implementation of this method in order to solve the type of large-scale, affine variational inequality problems which arise in the class of linearization techniques such as Newton's method [6]. Finally, the weakening of the assumption of monotonicity along the lines presented in [9] is a fruitful area for future research.

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